11. APPLICATIONS OF DERIVATIVES

1. Derivative as a Rate Measure

Rate of Change of Quantities

Let $y = f(x)$. Then, $\frac{dy}{dx}$ denotes the rate of change of y w.r.t. x and its value at $x = a$ is denoted by $\left[\frac{dy}{dx}\right]_{x=a}$.

If $x = f(t)$, $y = g(t)$ then by chain rule, we have

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \left(\frac{dy}{dt} \cdot \frac{dt}{dx}\right).$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the rate of change of the area of a circle with respect to its radius r when* $r = 6$ cm.

SOLUTION Let A be the area of a circle of radius r. Then,

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

$$\Rightarrow \left[\frac{dA}{dr}\right]_{r=6\text{ cm}} = (2\pi \times 6)\,\text{cm}^2/\text{cm} = (12\pi)\,\text{cm}^2/\text{cm}.$$

Hence, the area is changing at the rate of (12π) cm^2/cm.

EXAMPLE 2 *A stone is dropped into a quiet lake and the waves move in circles. If the radius of a circular wave increases at the rate of* 4 cm/sec, *find the rate of increase in its area at the instant when its radius is* 10 cm.

SOLUTION At any instant t, let the radius of the circle be r cm and its area be A cm^2. Then,

$$\frac{dr}{dt} = 4\,\text{cm/sec} \quad \text{(given)} \qquad \ldots \text{(i)}$$

$$\text{Now, } A = \pi r^2 \Rightarrow \frac{dA}{dt} = \left(\frac{dA}{dr} \cdot \frac{dr}{dt}\right)$$

$$= \frac{d}{dr}(\pi r^2) \cdot 4 \quad \left[\because A = \pi r^2 \text{ and } \frac{dr}{dt} = 4\right]$$

$$= (2\pi r \times 4)\ \text{cm}^2/\text{sec} = (8\pi r)\ \text{cm}^2/\text{sec}$$

$$\Rightarrow \quad \left[\frac{dA}{dt}\right]_{r=10} = (8\pi \times 10)\ \text{cm}^2/\text{sec} = (80\pi)\ \text{cm}^2/\text{sec}.$$

Hence, the area of the circle is increasing at the rate of $(80\pi)\ \text{cm}^2/\text{sec}$ at the instant when $r = 10$ cm.

EXAMPLE 3 *A spherical soap bubble is expanding so that its radius is increasing at the rate of* 0.02 cm/sec. *At what rate is the surface area increasing when its radius is* 5 cm? *(Take* $\pi = 3.14$.)

SOLUTION A soap bubble is in the form of a sphere. At an instant t, let its radius be r and surface area S. Then,

$$\frac{dr}{dt} = 0.02\ \text{cm/sec} \quad \text{(given)} \qquad \text{... (i)}$$

Now, $S = 4\pi r^2$

$$\Rightarrow \quad \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = (8 \times 3.14 \times r \times 0.02)\ \text{cm}^2/\text{sec}$$

$$\Rightarrow \quad \left[\frac{dS}{dt}\right]_{r=5} = (8 \times 3.14 \times 5 \times 0.02)\text{cm}^2/\text{sec} = 2.512\ \text{cm}^2/\text{sec}.$$

Hence, the surface area of the bubble is increasing at the rate of $2.512\ \text{cm}^2/\text{sec}$ at the instance when its radius is 5 cm.

EXAMPLE 4 *The volume of a spherical balloon is increasing at the rate of* $20\ \text{cm}^3/\text{sec}$. *Find the rate of change of its surface area at the instant when its radius is* 8 cm.

SOLUTION At any instant t, let r be the radius, V the volume and S the surface area of the balloon. Then,

$$\frac{dV}{dt} = 20\ \text{cm}^3/\text{sec} \quad \text{(given)} \qquad \text{... (i)}$$

Now, $V = \frac{4}{3}\pi r^3 \ \Rightarrow \ \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$$\Rightarrow \quad 20 = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad 20 = \frac{4}{3}\pi \times 3r^2 \times \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{5}{\pi r^2} \qquad \text{... (ii)}$$

$$\therefore \quad S = 4\pi r^2 \ \Rightarrow \ \frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$$

$$= \frac{d}{dr}(4\pi r^2) \cdot \frac{5}{\pi r^2}$$

$$= \left(8\pi r \times \frac{5}{\pi r^2}\right) = \frac{40}{r}$$

$$\Rightarrow \left[\frac{dS}{dt}\right]_{r=8\text{ cm}} = \left(\frac{40}{8}\right)\text{cm}^2/\text{sec} = 5\text{ cm}^2/\text{sec}.$$

Hence, the rate of change of surface area at the instant when $r = 8\,\text{cm}$ is $5\text{ cm}^2/\text{sec}$.

EXAMPLE 5 *The surface area of a spherical balloon is increasing at* $2\text{ cm}^2/\text{sec}$. *At what rate is the volume of the bubble increasing when the radius of the bubble is* 6 cm*?* **[CBSE 2005]**

SOLUTION At any instant t, let r be the radius, V the volume and S the surface area of the balloon. Then,

$$\frac{dS}{dt} = 2\text{ cm}^2/\text{sec} \quad \text{(given)} \qquad \text{... (i)}$$

$$\text{Now, } S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = \frac{dS}{dr}\cdot\frac{dr}{dt}$$

$$\Rightarrow 2 = \frac{d}{dr}(4\pi r^2)\cdot\frac{dr}{dt}$$

$$\Rightarrow 8\pi r\cdot\frac{dr}{dt} = 2$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r} \qquad \text{... (ii)}$$

$$\therefore\ V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\cdot\frac{dr}{dt} = 4\pi r^2\cdot\frac{dr}{dt}$$

$$= \left(4\pi r^2\cdot\frac{1}{4\pi r}\right) = r \quad \text{[using (ii)]}$$

$$\Rightarrow \left[\frac{dV}{dt}\right]_{r=6\text{ cm}} = 6\text{ cm}^3/\text{sec}.$$

EXAMPLE 6 *The volume of a cube is increasing at the rate of* $7\text{ cm}^3/\text{sec}$. *How fast is its surface area increasing at the instant when the length of an edge of the cube is* 12 cm*?* **[CBSE 2006C]**

SOLUTION At any instant t, let the length of each edge of the cube be x, V be its volume and S be its surface area. Then,

$$\frac{dV}{dt} = 7\text{ cm}^3/\text{sec} \quad \text{(given)} \qquad \text{... (i)}$$

$$\text{Now, } V = x^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dx}\cdot\frac{dx}{dt}$$

$$\Rightarrow 7 = \frac{d}{dx}(x^3)\cdot\frac{dx}{dt} \qquad [\because\ V = x^3]$$

$$\Rightarrow 3x^2\cdot\frac{dx}{dt} = 7$$

$$\Rightarrow \frac{dx}{dt} = \frac{7}{3x^2} \quad \text{... (ii)}$$

$$\therefore \quad S = 6x^2 \Rightarrow \frac{dS}{dt} = \frac{dS}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{d}{dx}(6x^2) \cdot \frac{7}{3x^2}$$

$$= \left(12x \times \frac{7}{3x^2}\right) = \frac{28}{x}$$

$$\Rightarrow \left[\frac{dS}{dt}\right]_{x=12} = \left(\frac{28}{12}\right) \text{cm}^2/\text{sec} = 2\frac{1}{3} \text{cm}^2/\text{sec}.$$

Hence, the surface area of the cube is increasing at the rate of $2\frac{1}{3}$ cm^2/sec at the instant when its edge is 12 cm.

EXAMPLE 7 *The length x of a rectangle is decreasing at the rate of* 5 cm/minute *and the width y is increasing at the rate of* 4 cm/minute. *When* $x = 8$ cm *and* $y = 6$ cm, *find the rate of change of (i) the perimeter, and (ii) the area of the rectangle.*

SOLUTION Given that $\frac{dx}{dt} = -5$ cm/minute and $\frac{dy}{dt} = 4$ cm/minute.

(i) Let P be the perimeter of the rectangle at any instant. Then,

$$P = 2(x + y) \Rightarrow \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5 + 4) \text{ cm/minute}$$

$$= -2 \text{ cm/minute}.$$

Hence, the perimeter of the rectangle is decreasing at the rate of 2 cm/minute.

(ii) Let A be the area of the rectangle at any instant. Then,

$$A = x \cdot y \Rightarrow \frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

$$= [(-5) \times 6 + 8 \times 4] \text{ cm}^2/\text{minute}$$

$$= 2 \text{ cm}^2/\text{minute}.$$

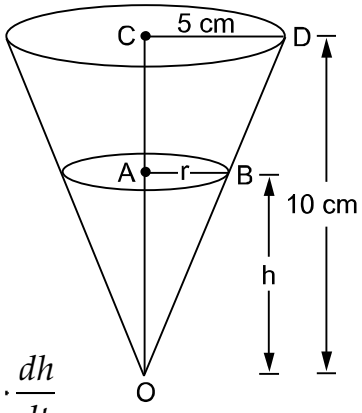
Hence, the area of the rectangle is increasing at the rate of 2 cm^2/minute.

EXAMPLE 8 *Water is leaking from a conical funnel at the rate of* 5 cm^3/sec. *If the radius of the base of the funnel is* 5 cm *and its altitude is* 10 cm, *find the rate at which the water level is dropping when it is* 2.5 cm *from the top.*

[CBSE 2004]

SOLUTION At any instant t, let r be the radius of the water level, h the height of the water level and V the volume of the water in the conical funnel. Then,

$$\frac{dV}{dt} = -5 \quad \text{(given)} \qquad \text{... (i)}$$



From similar $\triangle$s OAB and OCD, we have

$$\frac{AB}{OA} = \frac{CD}{OC} \Rightarrow \frac{r}{h} = \frac{5}{10} = \frac{1}{2} \Rightarrow r = \frac{1}{2}h.$$

Now, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times \left(\frac{1}{2}h\right)^2 \times h = \frac{1}{12}\pi h^3.$

$$\therefore \quad \frac{dV}{dt} = \left(\frac{dV}{dh} \times \frac{dh}{dt}\right) = \frac{d}{dh}\left(\frac{1}{12}\pi h^3\right) \cdot \frac{dh}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

$$\Rightarrow \quad -5 = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{-20}{\pi h^2} \qquad \text{... (i)}$$

$$\Rightarrow \quad \left(\frac{dh}{dt}\right)_{h=7.5 \text{ cm}} = \frac{-20}{\pi \times (7.5)^2} \qquad [\because \quad h = (10 - 2.5) \text{ cm} = 7.5 \text{ cm}]$$

$$= \frac{-16}{45\pi} \text{ cm/sec.}$$

Hence, the rate of change of water level at $h = 7.5$ cm is $\frac{-16}{45\pi}$ cm/sec.

EXAMPLE 9 *Sand is pouring from a pipe at the rate of* $12 \text{ cm}^3/\text{sec}$. *The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing, when the height is* 4 cm*?* **[CBSE 2011]**

SOLUTION At any instant t, let r be the radius, h the height and V the volume of the cone.

Then, $h = \frac{r}{6} \Rightarrow r = 6h.$

$$\therefore \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (6h)^2 h = 12\pi h^3.$$

$$\text{Now, } V = 12\pi h^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\Rightarrow 12 = \frac{d}{dh}(12\pi h^3) \times \frac{dh}{dt}$$

$$\Rightarrow 12 = 36\pi h^2 \times \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{3\pi h^2}$$

$$\Rightarrow \left(\frac{dh}{dt}\right)_{h=4 \text{ cm}} = \frac{1}{(3 \times \pi \times 4 \times 4)} \text{ cm/sec}$$

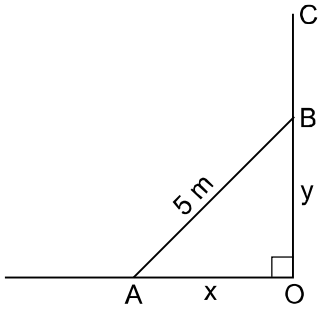
$$= \frac{1}{48\pi} \text{ cm/sec.}$$

Hence, the rate of increase of the height of the sand cone at the instant when $h = 4$ cm is $\frac{1}{48\pi}$ cm/sec.

EXAMPLE 10 *A* 5-m *long ladder is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall at the rate of* 2 m/sec. *How fast is its height on the wall decreasing when the foot of the ladder is* 4 m *away from the wall?*

SOLUTION Let OC be the wall. At a certain instant t, let AB be the position of the ladder such that $OA = x$ and $OB = y$.

Length of the ladder $AB = 5$ m.



Given that $\frac{dx}{dt} = 2$ m/sec.

From right $\triangle AOB$, we have

$$x^2 + y^2 = 25$$

$$\Rightarrow \quad 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \quad 2x \cdot 2 + 2y \cdot \frac{dy}{dt} = 0 \qquad \left[\because \quad \frac{dx}{dt} = 2\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-2x}{y} \qquad \text{... (i)}$$

Now, $x = 4 \Rightarrow y = \sqrt{5^2 - 4^2} = 3$.

Putting $x = 4$, $y = 3$ in (i) we get $\frac{dy}{dt} = \frac{-8}{3}$.

Hence, the required rate of decrease in the height of the ladder on the wall is (8/3) m/sec.

EXAMPLE 11 *The two equal sides of an isosceles triangle with fixed base* b cm *are decreasing at the rate of* 3 cm/sec. *How fast is the area decreasing when each of the equal sides is equal to the base?* **[CBSE 2006C]**

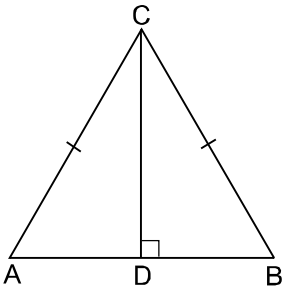
SOLUTION At a certain instant t, let $\triangle ABC$ be an isosceles triangle in which $AB = AC = x$ cm and $BC = b$ cm.

Then, $\frac{dx}{dt} = 3$ cm/sec (given) ... (i)

Draw $AD \perp BC$. Then, $BD = DC = \frac{b}{2}$.

$$\therefore \quad AD = \sqrt{x^2 - \frac{b^2}{4}} = \frac{\sqrt{4x^2 - b^2}}{2}$$

$$\therefore \quad A = \frac{1}{2} b \cdot \frac{\sqrt{4x^2 - b^2}}{2} = \frac{b\sqrt{4x^2 - b^2}}{4}$$



$$\Rightarrow \quad \frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$$

$$= \left(\frac{dA}{dx} \times 3 \right) \qquad \left[\because \quad \frac{dx}{dt} = 3 \text{ cm/sec} \right]$$

$$= \frac{d}{dx} \left\{ \frac{b\sqrt{4x^2 - b^2}}{4} \right\} \times 3$$

$$= \frac{3b}{4} \cdot \frac{1}{2} (4x^2 - b^2)^{-1/2} \cdot (8x) = \frac{3bx}{\sqrt{4x^2 - b^2}}.$$

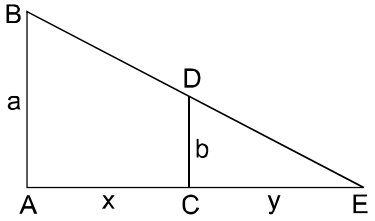
$$\therefore \quad \left[\frac{dA}{dt} \right]_{x=b} = \frac{3b^2}{\sqrt{4b^2 - b^2}} = \frac{3b^2}{\sqrt{3}\, b} = \sqrt{3}\, b.$$

EXAMPLE 12 *A point source of light along a straight road is at a height of 'a' metres. A boy 'b' metres in height is walking along the road. How fast is his shadow increasing if he is walking away from the light at the rate of 'c' metres per minute?* **[CBSE 2006C]**

SOLUTION Let AB be the lamp post, the lamp being at B. Then, $AB = a$ metres. At any instant t, let CD be the position of the boy and CE be his shadow. Then, $CD = b$ metres.

Let $AC = x$ metres and $CE = y$ metres.

Given that $\frac{dx}{dt} = c$ metres/min.



Clearly, $\triangle BAE \sim \triangle DCE$.

$$\therefore \quad \frac{AB}{CD} = \frac{AE}{CE}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{x + y}{y}$$

$$\Rightarrow \quad (a - b)y = bx$$

$$\Rightarrow \quad (a - b)\frac{dy}{dt} = b \cdot \frac{dx}{dt} = bc \qquad \left[\because \quad \frac{dx}{dt} = c \right]$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{bc}{(a - b)} \text{ m/min.}$$

Hence, the shadow is increasing at the rate of $\frac{bc}{(a-b)}$ m/min.

EXAMPLE 13 *A man* 160 cm *tall, walks away from a source of light situated at the top of a pole* 6 m *high, at the rate of* 1.1 m/s. *How fast is the length of his shadow increasing when he is* 1 m *away from the pole?*

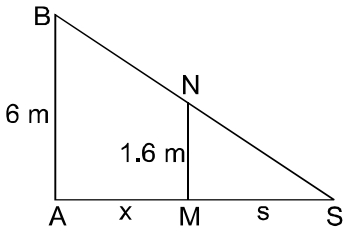
SOLUTION Let AB be the lamp post, the lamp being at B.

Then, $AB = 6$ m.

At any instance t, let MN be the position of the man and MS be his shadow. Then, $MN = 1.6$ m.

Let $AM = x$ metres and $MS = s$ metres.

Given that $\frac{dx}{dt} = 1.1$ m/s.



Clearly, $\triangle SAB \sim \triangle SMN$.

$\therefore \quad \frac{AS}{MS} = \frac{AB}{MN} = \frac{6}{1.6} = \frac{15}{4}$

$\Rightarrow \quad \frac{x+s}{s} = \frac{15}{4} \quad \Rightarrow \quad x = \frac{11}{4}s$, where $MS = s$.

$\Rightarrow \quad \frac{dx}{dt} = \frac{11}{4} \cdot \frac{ds}{dt} \quad \Rightarrow \quad 1.1 = \frac{11}{4} \cdot \frac{ds}{dt}$

$\Rightarrow \quad \frac{ds}{dt} = \left(\frac{1.1 \times 4}{11} \right) = 0.4$ m/s.

Hence, the length of the shadow is increasing at the rate of 0.4 m/s.

EXAMPLE 14 *A particle moves along the curve* $6y = x^3 + 2$. *Find the points on the curve at which the y-coordinate is changing* 8 *times as fast as the x-coordinate.* **[CBSE 2002]**

SOLUTION Let the required point be (x, y).

Given: $\frac{dy}{dt} = 8\frac{dx}{dt}$ ... (i)

Given curve is $6y = x^3 + 2$ ... (ii)

On differentiating both sides of (ii) w.r.t. t, we get

$$6\frac{dy}{dt} = 3x^2 \frac{dx}{dt}$$

$\Rightarrow \quad 6\left(8\frac{dx}{dt} \right) = 3x^2 \frac{dx}{dt}$ [using (i)]

$\Rightarrow \quad 3x^2 = 48 \Rightarrow \quad x^2 = 16 \Rightarrow \quad x = \pm 4.$

Putting $x = 4$ in (ii), we get $y = 11$.

Putting $x = -4$ in (ii), we get $y = \frac{-62}{6} = \frac{-31}{3}$.

Hence, the required points are (4, 11) and $\left(-4, \frac{-31}{3} \right)$.

EXAMPLE 15 *Find the points on the curve* $y^2 = 8x$ *for which the abscissa and ordinate change at the same rate.*

SOLUTION Let the required point be (x, y).

Given: $\frac{dy}{dt} = \frac{dx}{dt}$ ... (i)

Given curve is $y^2 = 8x$ ... (ii)

On differentiating both sides of (ii) w.r.t. t, we get

$$2y\frac{dy}{dt} = 8\frac{dx}{dt}$$

$\Rightarrow \; 2y \cdot \frac{dx}{dt} = 8\frac{dx}{dt}$ [using (i)]

$\Rightarrow \; 2y = 8 \; \Rightarrow \; y = 4.$

Putting $y = 4$ in (ii), we get $8x = 16$ and therefore, $x = 2$.

Hence, the required point is (2, 4).

EXAMPLE 16 *At what points of the ellipse* $16x^2 + 9y^2 = 400$ *does the ordinate decrease at the same rate at which the abscissa increases?*

SOLUTION Let the required point be (x, y). Then,

$$\frac{dy}{dt} = -\frac{dx}{dt}$$ ... (i)

Given curve is $16x^2 + 9y^2 = 400$ ... (ii)

On differentiating both sides of (ii) w.r.t. t, we get

$$32x \cdot \frac{dx}{dt} + 18y \cdot \frac{dy}{dt} = 0$$

$\Rightarrow \; 32x \cdot \frac{dx}{dt} + 18y \cdot \left(-\frac{dx}{dt}\right) = 0$ [using (i)]

$\Rightarrow \; (32x - 18y) = 0 \; \Rightarrow \; y = \frac{16}{9}x.$

Putting $y = \frac{16}{9}x$ in (ii), we get

$$16x^2 + 9 \times \left(\frac{16x}{9}\right)^2 = 400 \; \Rightarrow \; x^2 = 9 \; \Rightarrow \; x = \pm 3.$$

Now, $x = 3 \; \Rightarrow \; y = \left(\frac{16}{9} \times 3\right) = \frac{16}{3}$,

$$x = -3 \; \Rightarrow \; y = \left[\frac{16}{9} \times (-3)\right] = \frac{-16}{3}.$$

Hence, the required points are $\left(3, \frac{16}{3}\right)$ and $\left(-3, \frac{-16}{3}\right)$.

MARGINAL COST AND MARGINAL REVENUE

Marginal Cost: Let C be the total cost of producing and marketing x units of a product. Then, the marginal cost (MC) is defined as, $\boldsymbol{MC = \frac{dC}{dx}}$.

Marginal Revenue: The rate of change of total revenue with respect to the quantity sold is called the marginal revenue (MR) and therefore, $\boldsymbol{MR = \frac{dR}{dx}}$.

EXAMPLE 17 *The total cost $C(x)$ of producing x items in a firm is given by*

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 6000.$$

Find the marginal cost when 4 units are produced.

SOLUTION Given: $C(x) = 0.005x^3 - 0.02x^2 + 30x + 6000$

$$\Rightarrow MC = \frac{dC}{dx}$$

$$= \frac{d}{dx}(0.005x^3 - 0.02x^2 + 30x + 6000)$$

$$= \{(0.005 \times 3x^2) - (0.02 \times 2x) + 30\}$$

$$\Rightarrow [MC]_{x=4} = \{(0.005 \times 3 \times 4^2) - (0.02 \times 2 \times 4) + 30\}$$

$$= (0.24 - 0.16 + 30) = 30.08.$$

Hence, the required marginal cost is ₹ 30.08.

EXAMPLE 18 *The total revenue received from the sale of x units of a product is given by*

$$R(x) = 3x^2 + 40x + 10.$$

Find the marginal revenue when $x = 5$.

SOLUTION Given: $R(x) = 3x^2 + 40x + 10$

$$\Rightarrow \quad MR = \frac{dR}{dx}$$

$$= \frac{d}{dx}(3x^2 + 40x + 10) = 6x + 40$$

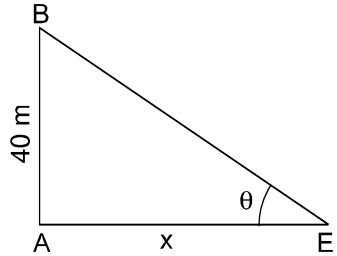
$$\Rightarrow \quad [MR]_{x=5} = (6 \times 5 + 40) = 70.$$

Hence, the required marginal revenue is ₹ 70.

EXERCISE 11A

1. The side of a square is increasing at the rate of 0.2 cm/s. Find the rate of increase of the perimeter of the square.
2. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?
3. The radius of a circle is increasing uniformly at the rate of 0.3 centimetre per second. At what rate is the area increasing when the radius is 10 cm? (Take $\pi = 3.14$.)
4. The side of a square sheet of metal is increasing at 3 centimetres per minute. At what rate is the area increasing when the side is 10 cm long?
5. The radius of a circular soap bubble is increasing at the rate of 0.2 cm/s. Find the rate of increase of its surface area when the radius is 7 cm.

6. The radius of an air bubble is increasing at the rate of 0.5 centimetre per second. At what rate is the volume of the bubble increasing when the radius is 1 centimetre?
7. The volume of a spherical balloon is increasing at the rate of 25 cubic centimetres per second. Find the rate of change of its surface at the instant when its radius is 5 cm.
8. A balloon which always remains spherical is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm.
9. The bottom of a rectangular swimming tank is 25 m by 40 m. Water is pumped into the tank at the rate of 500 cubic metres per minute. Find the rate at which the level of water in the tank is rising.
10. A stone is dropped into a quiet lake and waves move in circles at a speed of 3.5 cm per second. At the instant when the radius of the circular wave is 7.5 cm, how fast is the enclosed area increasing? (Take $\pi = 22/7$.)
11. A 2-m tall man walks at a uniform speed of 5 km per hour away from a 6-metre-high lamp post. Find the rate at which the length of his shadow increases.
12. An inverted cone has a depth of 40 cm and a base of radius 5 cm. Water is poured into it at a rate of 1.5 cubic centimetres per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.
13. Sand is pouring from a pipe at the rate of 18 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is one-sixth of the radius of the base. How fast is the height of the sand cone increasing when its height is 3 cm?
14. Water is dripping through a tiny hole at the vertex in the bottom of a conical funnel at a uniform rate of 4 cm^3/s. When the slant height of the water is 3 cm, find the rate of decrease of the slant height of the water, given that the vertical angle of the funnel is 120°.
15. Oil is leaking at the rate of 16 mL/s from a vertically kept cylindrical drum containing oil. If the radius of the drum is 7 cm and its height is 60 cm, find the rate at which the level of the oil is changing when the oil level is 18 cm.
16. A 13-m long ladder is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 5 m away from the wall?
17. A man is moving away from a 40-m high tower at a speed of 2 m/s. Find the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 30 metres from the foot of the tower. Assume that the eye level of the man is 1.6 m from the ground.



18. Find an angle x which increases twice as fast as its sine.
19. The radius of a balloon is increasing at the rate of 10 cm/s. At what rate is the surface area of the balloon increasing when the radius is 15 cm?
20. An edge of a variable cube is increasing at the rate of 5 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?
21. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area is increasing when the side is 10 cm.

[CBSE 2014C]

ANSWERS (EXERCISE 11A)

1. $\frac{dP}{dt} = 0.8\,\text{cm/s}$ **2.** $\frac{dC}{dt} = 4.4\,\text{cm/s}$ **3.** $\frac{dA}{dt} = 18.84\ \text{cm}^2/\text{s}$

4. $\frac{dA}{dt} = 60\ \text{cm}^2/\text{min}$ **5.** $\frac{dS}{dt} = 35.2\ \text{cm}^2/\text{s}$ **6.** $\frac{dV}{dt} = 6.28\ \text{cm}^3/\text{s}$

7. $\frac{dS}{dt} = 10\ \text{cm}^2/\text{s}$ **8.** $\frac{dr}{dt} = 0.32\,\text{cm/s}$ **9.** $\frac{dh}{dt} = 0.5\ \text{m/min}$

10. $165\ \text{cm}^2/\text{s}$ **11.** $2.5\ \text{km/h}$ **12.** $\frac{1}{10\pi}\,\text{cm/s}$

13. $\frac{1}{18\pi}\,\text{cm/s}$ **14.** $\frac{32}{27\pi}\,\text{cm/s}$ **15.** $\frac{16}{49\pi}\,\text{cm/s}$

16. $\frac{5}{6}\,\text{m/s}$ **17.** 0.032 radian/second **18.** $\frac{\pi}{3}$

19. $1200\pi\ \text{cm}^2/\text{s}$ **20.** $1500\ \text{cm}^3/\text{s}$ **21.** $10\sqrt{3}\ \text{cm}^2/\text{s}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11A)

5. $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = (8\pi r)\cdot\frac{dr}{dt} = (8\pi r)(0.2)$

$\Rightarrow \left[\frac{dS}{dt}\right]_{r=7} = (1.6\pi \times 7)\ \text{cm}^2/\text{s}.$

6. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = (4\pi r^2)\cdot\frac{dr}{dt} = (4\pi r^2)\times 0.5.$

7. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = (4\pi r^2)\cdot\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{25}{4\pi r^2}\cdot$

$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r\cdot\frac{dr}{dt} = \left\{8\pi r\cdot\frac{25}{4\pi r^2}\right\} = \frac{50}{r}$

$\Rightarrow \left[\frac{dS}{dt}\right]_{r=5} = \left(\frac{50}{5}\right)\text{cm}^2/\text{s}.$

8. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2\cdot\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$

$\Rightarrow \left[\frac{dr}{dt}\right]_{r=15} = \frac{225}{\pi\times(15)^2} = \frac{1}{\pi}\,\text{cm/s}.$

9. $V = (25 \times 40 \times h) \Rightarrow \dfrac{dV}{dt} = \left(1000 \times \dfrac{dh}{dt}\right)$

$\Rightarrow 500 = \left(1000 \times \dfrac{dh}{dt}\right) \Rightarrow \dfrac{dh}{dt} = 0.5 \text{ m/min.}$

10. $A = \pi r^2 \Rightarrow \dfrac{dA}{dt} = 2\pi r \cdot \dfrac{dr}{dt} = (2\pi r \times 3.5) = 7\pi r$

$\Rightarrow \left[\dfrac{dA}{dt}\right]_{r=7.5} = (7\pi \times 7.5)\ \text{cm}^2/\text{s} = 165\ \text{cm}^2/\text{s.}$

13. $h = \dfrac{r}{6} \Rightarrow r = 6h.$ So, $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi(6h)^2 h = 12\pi h^3.$

$\therefore \dfrac{dV}{dt} = 36\pi h^2 \cdot \dfrac{dh}{dt} \Rightarrow 18 = 36\pi h^2 \cdot \dfrac{dh}{dt} \Rightarrow \dfrac{dh}{dt} = \dfrac{18}{36\pi h^2} = \dfrac{1}{2\pi h^2}$

$\Rightarrow \left[\dfrac{dh}{dt}\right]_{h=3} = \dfrac{1}{(2\pi \times 9)}\ \text{cm/s.}$

14. $V = \dfrac{1}{3}\pi(l \sin 60^\circ)^2 (l \cos 60^\circ) = \dfrac{\pi}{8} l^3.$

$\therefore \dfrac{dV}{dt} = \dfrac{3\pi l^2}{8} \cdot \dfrac{dl}{dt} \Rightarrow \dfrac{3\pi l^2}{8} \cdot \dfrac{dl}{dt} = -4 \Rightarrow \dfrac{dl}{dt} = \dfrac{-32}{3\pi l^2} \Rightarrow \left[\dfrac{dl}{dt}\right]_{l=3} = \dfrac{-32}{27\pi}\ \text{cm/s.}$

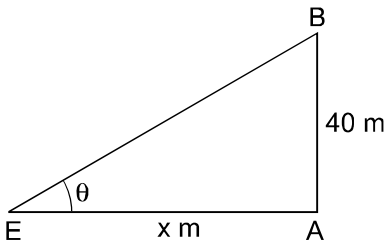
15. $V = (\pi \times 7 \times 7 \times h) \Rightarrow \dfrac{dV}{dt} = 49\pi \cdot \dfrac{dh}{dt} \Rightarrow 49\pi \cdot \dfrac{dh}{dt} = 16 \Rightarrow \dfrac{dh}{dt} = \dfrac{16}{49\pi}\ \text{cm/s.}$

17. Let at any time t, the man be at a distance of x metres from tower AB and let θ be the angle of elevation at that time.

Then, $x = 40 \cot\theta \Rightarrow \dfrac{dx}{dt} = -40\,\text{cosec}^2\theta \cdot \dfrac{d\theta}{dt}.$

$\therefore \dfrac{d\theta}{dt} = \dfrac{-2\sin^2\theta}{40} = -\dfrac{1}{20} \cdot \left(\dfrac{AB}{BE}\right)^2 = -\dfrac{1}{20} \cdot \dfrac{(40)^2}{(40)^2 + x^2}.$

Find its value when $x = 30$ m.



18. $\dfrac{dx}{dt} = 2 \cdot \dfrac{d}{dt}(\sin x) = 2\cos x \cdot \dfrac{dx}{dt}.$ So, $\cos x = \dfrac{1}{2}$ or $x = \dfrac{\pi}{3}.$

19. $S = 4\pi r^2 \Rightarrow \dfrac{dS}{dt} = 8\pi r \cdot \dfrac{dr}{dt} = (8\pi \times 15 \times 10)\ \text{cm}^2/\text{s.}$

20. $V = x^3 \Rightarrow \dfrac{dV}{dt} = 3x^2 \cdot \dfrac{dx}{dt} = [3 \times (10)^2 \times 5]\ \text{cm}^3/\text{s.}$

21. $A = \dfrac{\sqrt{3}}{4}a^2 \Rightarrow \dfrac{dA}{da} = \dfrac{\sqrt{3}}{2}a \Rightarrow \dfrac{dA}{dt} \cdot \dfrac{dt}{da} = \dfrac{\sqrt{3}}{2}a$

$\Rightarrow \dfrac{dA}{dt} \times \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}a \Rightarrow \dfrac{dA}{dt} = \sqrt{3}a \Rightarrow \left[\dfrac{dA}{dt}\right]_{a=10} = \sqrt{3} \times 10\ \text{cm}^2.$

2. Errors and Approximation

Let $y = f(x)$. Then, $\lim\limits_{\delta x \to 0} \dfrac{f(x + \delta x) - f(x)}{\delta x} = f'(x).$

$\therefore \dfrac{f(x + \delta x) - f(x)}{\delta x} = f'(x) + \epsilon$, where $\epsilon \to 0$ when $\delta x \to 0$

$\Rightarrow \quad f(x+\delta x)-f(x)=f'(x)\cdot\delta x+\in\cdot\delta x$

$\Rightarrow \quad f(x+\delta x)-f(x)=f'(x)\cdot\delta x \qquad$ (approximately)

$\Rightarrow \quad \delta y=f'(x)\cdot\delta x \qquad [\because \quad f(x+\delta x)-f(x)=\delta y].$

Thus, if δx is an error in x then the corresponding error in y is δy. These small values δx and δy are called ***differentials.***

(i) ***Absolute Error***: δx is called an absolute error in x.

(ii) ***Relative Error***: $\dfrac{\delta x}{x}$ is called the relative error.

(iii) ***Percentage Error***: $\left(\dfrac{\delta x}{x}\times 100\right)$ is called the percentage error.

SOLVED EXAMPLES

EXAMPLE 1 *Using differentials, find the approximate value of* $(82)^{1/4}$ *up to three places of decimal.* **[CBSE 2005]**

SOLUTION Let $f(x)=x^{1/4}$. Then, $f'(x)=\dfrac{1}{4x^{3/4}}\cdot$

Now, $\{f(x+\delta x)-f(x)\}=f'(x)\cdot\delta x$

$\Rightarrow \quad \{f(x+\delta x)-f(x)\}=\dfrac{1}{4x^{3/4}}\cdot\delta x \qquad$ ... (i)

We may write, $82=(81+1)$.

Putting $x=81$ and $\delta x=1$ in (i), we get

$$f(81+1)-f(81)=\frac{1}{4\times(81)^{3/4}}\cdot 1$$

$\Rightarrow \quad f(82)-f(81)=\dfrac{1}{(4\times 3^3)}=\dfrac{1}{108}$

$\Rightarrow \quad f(82)=\left\{f(81)+\dfrac{1}{108}\right\}=\left\{(81)^{1/4}+\dfrac{1}{108}\right\}=3+0.009=3.009.$

EXAMPLE 2 *Find the approximate value of the cube root of* 127.

SOLUTION Let $f(x)=x^{1/3}$. Then, $f'(x)=\dfrac{1}{3x^{2/3}}\cdot$

Now, $\{f(x+\delta x)-f(x)\}=f'(x)\cdot\delta x$

$\Rightarrow \quad \{f(x+\delta x)-f(x)\}=\dfrac{1}{3x^{2/3}}\cdot\delta x \qquad$ ... (i)

We may write, $127 = (125 + 2)$.

Putting $x=125$ and $\delta x=2$ in (i), we get

$$f(125+2)-f(125)=\frac{1}{3\times(125)^{2/3}}\times 2$$

$$\Rightarrow \quad f(127) - f(125) = \frac{2}{75}$$

$$\Rightarrow \quad f(127) = f(125) + \frac{2}{75} = \left\{(125)^{1/3} + \frac{2}{75}\right\} = \left(5 + \frac{2}{75}\right) = \frac{377}{75}$$

$$\Rightarrow \quad \sqrt[3]{127} = \frac{377}{75} = 5.026.$$

EXAMPLE 3 *Using differentials find the approximate value of the square root of* 26. **[CBSE 2000]**

SOLUTION Let $f(x) = \sqrt{x}$. Then, $f'(x) = \frac{1}{2\sqrt{x}}$.

Now, $\{f(x + \delta x) - f(x)\} = f'(x) \cdot \delta x$

$$\Rightarrow \quad \{f(x + \delta x) - f(x)\} = \frac{1}{2\sqrt{x}} \cdot \delta x \qquad \text{... (i)}$$

We may write, $26 = (25 + 1)$.

Putting $x = 25$ and $\delta x = 1$ in (i), we get

$$f(26) - f(25) = \frac{1}{2\sqrt{25}} \times 1$$

$$\Rightarrow \quad f(26) = f(25) + \frac{1}{10} = \left(\sqrt{25} + \frac{1}{10}\right) = \left(5 + \frac{1}{10}\right) = (5 + 0.1) = 5.1$$

$$\Rightarrow \quad \sqrt{26} = 5.1.$$

Hence, $\sqrt{26} = 5.1$.

EXAMPLE 4 *Using differentials find the approximate value of* $\sqrt{0.037}$. **[CBSE 2005C]**

SOLUTION Let $f(x) = \sqrt{x}$. Then, $f'(x) = \frac{1}{2\sqrt{x}}$.

Now, $\{f(x + \delta x) - f(x)\} = f'(x) \cdot \delta x$

$$\Rightarrow \quad \{f(x + \delta x) - f(x)\} = \frac{1}{2\sqrt{x}} \cdot \delta x \qquad \text{... (i)}$$

We may write, $0.037 = (0.04 - 0.003)$.

Putting $x = 0.04$ and $\delta x = -0.003$ in (i), we get

$$f(0.04 - 0.003) - f(0.04) = \frac{1}{2\sqrt{0.04}} \times (-0.003)$$

$$\Rightarrow \quad f(0.037) = f(0.04) - \frac{0.003}{0.4}$$

$$\Rightarrow \quad \sqrt{0.037} = \left(\sqrt{0.04} - \frac{3}{400}\right) = \left(0.2 - \frac{3}{400}\right) = \frac{77}{400}$$

$$\Rightarrow \quad \sqrt{0.037} = 0.1925.$$

EXAMPLE 5 *Find the approximate value of* $\tan 46°$, *it is being given that* $1° = 0.01745$ *radian.*

SOLUTION Let $f(x) = \tan x$. Then, $f'(x) = \sec^2 x$.

Now, $f(x + \delta x) - f(x) = f'(x) \cdot \delta x$

$\Rightarrow \quad f(x + \delta x) - f(x) = \sec^2 x \cdot \delta x$ ... (i)

Putting $x = 45°$, $\delta x = 1° = 0.01745$ in (i), we get

$$f(46°) - f(45°) = (\sec^2 45°) \times 0.01745$$

$\Rightarrow \quad \tan 46° - \tan 45° = (\sec^2 45°) \times 0.01745$

$\Rightarrow \quad \tan 46° = \tan 45° + (\sec^2 45°) \times 0.01745$

$$= (1 + 2 \times 0.01745) = 1.03490.$$

Hence, $\tan 46° = 1.03490$.

EXAMPLE 6 *Find the approximate value of* $\log_{10} 10.1$, *it being given that* $\log_{10} e = 0.4343$.

SOLUTION Let $f(x) = \log_{10} x$. Then, $f'(x) = \frac{1}{x}(\log_{10} e)$.

Now, $f(x + \delta x) - f(x) = f'(x) \cdot \delta x$

$\Rightarrow \quad f(x + \delta x) - f(x) = \frac{1}{x}(\log_{10} e) \cdot \delta x$

$\Rightarrow \quad f(x + \delta x) - f(x) = \frac{0.4343}{x} \cdot \delta x$ ... (i)

Putting $x = 10$ and $\delta x = 0.1$ in (i), we get

$$f(10.1) - f(10) = \frac{0.4343}{10} \times 0.1$$

$\Rightarrow \quad \log_{10}(10.1) - \log_{10} 10 = 0.004343$

$\Rightarrow \quad \log_{10}(10.1) = (\log_{10} 10) + 0.004343$

$$= (1 + 0.004343) = 1.004343.$$

Hence, $\log_{10}(10.1) = 1.004343$.

EXAMPLE 7 *If* $f(x) = 3x^2 + 15x + 5$, *then find the approximate value of* $f(3.02)$, *using differentials.* **[CBSE 2008C, '14]**

SOLUTION $f(x) = 3x^2 + 15x + 5 \Rightarrow f'(x) = 6x + 15$

Now, $f(x + \delta x) - f(x) = f'(x) \cdot \delta x$

$\Rightarrow \quad f(x + \delta x) - f(x) = (6x + 15) \cdot \delta x$ ... (i)

Putting $x = 3$ and $\delta x = 0.02$ in (i), we get

$$f(3.02) - f(3) = (6 \times 3 + 15) \times (0.02)$$

$\Rightarrow \quad f(3.02) - 77 = 0.66 \qquad [\because \quad f(3) = 3 \times 3^2 + 15 \times 3 + 5 = 77]$

$\Rightarrow \quad f(3.02) = 77 + 0.66 = 77.66.$

Use of the Formula: $\delta y = \dfrac{dy}{dx} \cdot \delta x$

EXAMPLE 8 *If the radius of a circle increases from* 5 cm *to* 5.1 cm, *find the increase in area.*

SOLUTION Area of a circle of radius r is given by $A = \pi r^2$.

$$\text{Now, } A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$$

$$\Rightarrow \left[\frac{dA}{dr}\right]_{r=5} = (2\pi \times 5)\ \text{cm}^2 = 10\pi\ \text{cm}^2.$$

$$\text{Also, } \delta r = (5.1 - 5)\ \text{cm} = 0.1\ \text{cm}.$$

$$\therefore\ \delta A = \frac{dA}{dr} \cdot \delta r = (10\pi \times 0.1)\ \text{cm}^2 = \pi\ \text{cm}^2.$$

Hence, the increase in area is $\pi\ \text{cm}^2$.

EXAMPLE 9 *If* $y = (x^4 - 12)$ *and if* x *changes from* 2 *to* 1.99, *what is the approximate change in* y? **[CBSE 2002]**

SOLUTION $y = (x^4 - 12) \Rightarrow \dfrac{dy}{dx} = 4x^3$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=2} = (4 \times 2^3) = 32.$$

$$\text{Let } \delta x = (1.99 - 2) = -0.01.$$

$$\therefore\ \delta y = \frac{dy}{dx} \cdot \delta x = 32 \times (-0.01) = -0.32.$$

EXAMPLE 10 *The time* T *of oscillation of a simple pendulum of length* l *is given by* $T = 2\pi\sqrt{\dfrac{l}{g}}$. *Find the percentage error in* T, *corresponding to an error of* 2% *in the value of* l.

SOLUTION $T = 2\pi\sqrt{\dfrac{l}{g}} \Rightarrow \log T = \log 2 + \log \pi + \dfrac{1}{2}\log l - \dfrac{1}{2}\log g$

$$\Rightarrow \frac{1}{T} \cdot \frac{dT}{dl} = \frac{1}{2l} \Rightarrow \frac{1}{T} \cdot \frac{dT}{dl} \cdot \delta l = \frac{1}{2l}\delta l$$

$$\Rightarrow \frac{1}{T} \cdot \delta T = \frac{1}{2l} \cdot \delta l \quad [\because\ \delta T = \frac{dT}{dl} \cdot \delta l\ ]$$

$$\Rightarrow \left(\frac{\delta T}{T} \times 100\right) = \frac{1}{2} \cdot \left(\frac{\delta l}{l} \times 100\right) = \frac{1}{2} \times 2 = 1.$$

$\therefore$ percentage error in $T = 1\%$.

EXAMPLE 11 *If the error committed in measuring the radius of a circle be* 0.01%, *find the corresponding error in calculating the area.*

SOLUTION $A = \pi r^2 \Rightarrow \dfrac{dA}{dr} = 2\pi r \Rightarrow \dfrac{dA}{dr} \cdot \delta r = 2\pi r \cdot \delta r$

$\Rightarrow \delta A = 2\pi r^2 \cdot \dfrac{\delta r}{r} \quad [\because \dfrac{dA}{dr} \cdot \delta r = \delta A]$

$\Rightarrow \dfrac{\delta A}{A} = 2 \cdot \dfrac{\delta r}{r} \quad [\because A = \pi r^2]$

$\Rightarrow \left(\dfrac{\delta A}{A} \times 100\right) = 2 \cdot \left(\dfrac{\delta r}{r} \times 100\right)$

$\Rightarrow$ percentage error in area $= (2 \times 0.01) = 0.02\%$.

Hence, the percentage error in area $= 0.02\%$.

EXAMPLE 12 *If in a triangle ABC, the side c and the angle C remain constant while the remaining elements are changed slightly, show that* $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} = 0$.

SOLUTION As given, we have $\dfrac{c}{\sin C} = k$ (constant).

$\therefore \quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = k \Rightarrow a = k \sin A$ and $b = k \sin B$.

$\therefore \quad da = k \cos A \cdot dA$ and $db = k \cos B \cdot dB$

or $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} = k(dA + dB) = kd(A + B) = kd(\pi - C) = 0$.

Hence, $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} = 0$.

EXAMPLE 13 *The area S of a triangle is calculated by measuring the sides b and c, and* $\angle A$. *If there be an error* δA *in the measurement of* $\angle A$, *show that the relative error in area is given by*

$$\frac{\delta S}{S} = \cot A \cdot \delta A.$$

SOLUTION $S = \dfrac{1}{2} bc \sin A \Rightarrow \dfrac{dS}{dA} = \dfrac{1}{2} bc \cos A \Rightarrow \dfrac{dS}{dA} \cdot \delta A = \dfrac{1}{2} bc \cos A \cdot \delta A$

$\Rightarrow \delta S = \dfrac{1}{2} bc \cos A \cdot \delta A \Rightarrow \delta S = \dfrac{1}{2} bc \sin A \cdot (\cot A) \cdot \delta A$

$\Rightarrow \delta S = S \cdot (\cot A) \cdot \delta A \Rightarrow \dfrac{\delta S}{S} = (\cot A) \cdot \delta A$.

Hence, $\dfrac{\delta S}{S} = (\cot A) \cdot \delta A$.

EXERCISE 11B

Using differentials, find the approximate values of:

1. $\sqrt{37}$ **[CBSE 2000]** 2. $\sqrt[3]{29}$

3. $\sqrt[3]{27}$ 4. $\sqrt{0.24}$ **[CBSE 2001]**

5. $\sqrt{49.5}$ **[CBSE 2012]** 6. $\sqrt[4]{15}$ 7. $\dfrac{1}{(2.002)^2}$

8. $\log_e 10.02$, given that $\log_e 10 = 2.3026$

9. $\log_{10}(4.04)$, it being given that $\log_{10} 4 = 0.6021$ and $\log_{10} e = 0.4343$

10. $\cos 61°$, it being given that $\sin 60° = 0.86603$ and $1° = 0.01745$ radian

11. If $y = \sin x$ and x changes from $\dfrac{\pi}{2}$ to $\dfrac{22}{14}$, what is the approximate change in y?

12. A circular metal plate expands under heating so that its radius increases by 2%. Find the approximate increase in the area of the plate, if the radius of the plate before heating is 10 cm.

13. If the length of a simple pendulum is decreased by 2%, find the percentage decrease in its period T, where $T = 2\pi\sqrt{\dfrac{l}{g}}$.

14. The pressure p and the volume V of a gas are connected by the relation, $pV^{1/4} = k$, where k is a constant. Find the percentage increase in the pressure, corresponding to a diminution of 0.5% in the volume.

15. The radius of a sphere shrinks from 10 cm to 9.8 cm. Find approximately the decrease in (i) volume, and (ii) surface area.

16. If there is an error of 0.1% in the measurement of the radius of a sphere, find approximately the percentage error in the calculation of the volume of the sphere.

17. Show that the relative error in the volume of a sphere, due to an error in measuring the diameter, is three times the relative error in the diameter.

ANSWERS (EXERCISE 11B)

1. 6.08 **2.** 3.074 **3.** 5.027 **4.** 0.49

5. 7.0357 **6.** $\dfrac{63}{32}$ **7.** 0.2495 **8.** 2.3046

9. 0.606443 **10.** 0.4849 **11.** No change **12.** $\dfrac{2\pi}{5}$ cm^2

13. 1% **14.** 0.125% **15.** (i) 80π cm^3 (ii) 16π cm^2

16. 0.3%

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11B)

7. Let $f(x) = \frac{1}{x^2}$. Then $f(x + \delta x) - f(x) = \frac{-2}{x^3} \cdot \delta x$.

   Take $x = 2$ and $\delta x = 0.002$.

12. $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r = 20\pi$, when $r = 10$.

$$\therefore \quad \delta A = \frac{dA}{dr} \cdot \delta r = \left(20\pi \times \frac{2}{100} \right) = \frac{2\pi}{5}.$$

13. $\log T = \log\left(\frac{2\pi}{\sqrt{g}}\right) + \frac{1}{2}\log l \Rightarrow \frac{1}{T} \cdot \frac{dT}{dl} = \frac{1}{2l}$.

$$\therefore \quad \delta T = \frac{T}{2l} \cdot \delta l \Rightarrow \left(\frac{\delta T}{T} \times 100 \right) = \left(\frac{1}{2} \cdot \frac{\delta l}{l} \times 100 \right) = \left\{ \frac{1}{2} \times \left(\frac{-2}{100} \right) \times 100 \right\} = -1.$$

14. $pV^{1/4} = k \Rightarrow \log p + \frac{1}{4}\log V = \log k$

$$\Rightarrow \frac{1}{p} \cdot \frac{dp}{dV} + \frac{1}{4V} = 0 \Rightarrow \frac{1}{p} \cdot \frac{dp}{dV} \cdot \delta V = -\frac{1}{4V} \cdot \delta V$$

$$\Rightarrow \frac{1}{p} \cdot \delta p = -\frac{1}{4V} \cdot \delta V \Rightarrow \left(\frac{\delta p}{p} \times 100 \right)$$

$$\Rightarrow \left(\frac{\delta p}{p} \times 100 \right) = -\frac{1}{4}\left(\frac{\delta V}{V} \times 100 \right) = \left(-\frac{1}{4} \right)\left(-\frac{1}{2} \right) = \frac{1}{8}.$$

3. Rolle's and Lagrange's Theorems

Rolle's Theorem

Let f be a real-valued function, defined in the closed interval $[a, b]$ *such that*

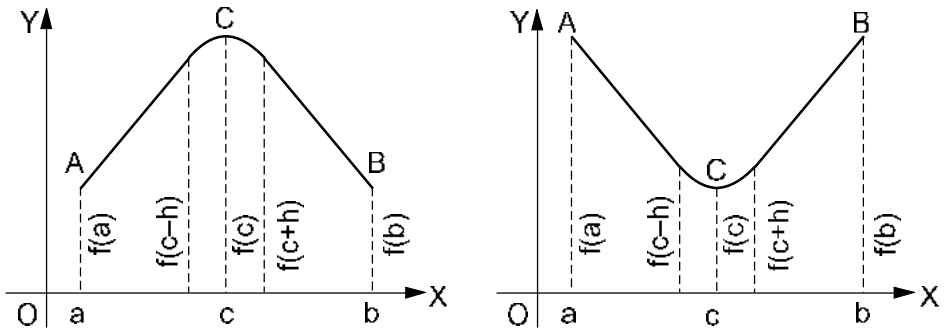
(i) f *is continuous on* $[a, b]$; *(ii)* f *is differentiable on* $]a, b[$;

(iii) $f(a) = f(b)$.

Then, there exists a real number c in the open interval $]a, b[$ *such that* $f'(c) = 0$.

PROOF If $f(x) = c$, where c is a constant then $f'(x) = 0$ for all $x \in]a, b[$.
So, in this case, the theorem follows.

Now, consider the case, when f is not a constant function. Since $f(a) = f(b)$, the function should either increase or decrease, when x takes values slightly greater than a.



Let $f(x)$ be increasing at $x > a$.

Since $f(b) = f(a)$, $f(x)$ must cease to increase and begin to decrease at some point $x = c \in]a,\ b[$.

Clearly, at such a point, the function has a maximum value. Thus, for a very small and positive value of h, we have

$$f(c+h) - f(c) < 0 \text{ and } f(c-h) - f(c) < 0.$$

$$\therefore \quad \frac{f(c+h)-f(c)}{h} < 0 \text{ and } \frac{f(c-h)-f(c)}{-h} > 0.$$

So, $$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \le 0 \text{ and } \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h} \ge 0.$$

But, if $$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \ne \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h}$$ then

$Rf'(c) \ne Lf'(c)$ and this would imply that $f(x)$ is not differentiable at $x = c$, contradicting the given hypothesis.

$$\therefore \quad \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h} = 0.$$

Hence, $f'(c) = 0$, where $a < c < b$.

Similarly, the theorem can be proved for the case when $f(x)$ is initially decreasing for values of $x > a$, and it ceases to decrease at $x = c$.

Hence, there exists $c \in]a,\ b[$ such that $f'(c) = 0$.

REMARK Rolle's theorem is applicable to a function only when all the three properties above listed are satisfied by it. If even a single property is violated, we say that the theorem is not applicable on such a function.

GEOMETRICAL SIGNIFICANCE OF ROLLE'S THEOREM Let f be a real function defined on $[a, b]$ and let Rolle's theorem be applicable on it. Then, f being continuous on $[a, b]$, it follows that we can draw a graph of $f(x)$ between the values $x = a$ and $x = b$.

Also, $f(x)$ being differentiable in $]a,\ b[$, it follows that the graph of $f(x)$ has a tangent at each point of $]a,\ b[$.

Now, the existence of a real number $c \in]a,\ b[$ such that $f'(c) = 0$ shows that the tangent to the curve at $x = c$ has a slope 0, i.e., it is parallel to the x-axis.

SOLVED EXAMPLES

EXAMPLE 1 *Verify Rolle's theorem for the function* $f(x) = x^3 - 6x^2 + 11x - 6$ *in the interval* $[1, 3]$.

SOLUTION Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[1, 3]$.

(ii) $f'(x) = 3x^2 - 12x + 11$, which clearly exists for all values of $x \in [1, 3]$.

So, $f(x)$ is differentiable on the open interval $]1, 3[$.

(iii) $f(1) = (1^3 - 6 \times 1^2 + 11 \times 1 - 6) = 0$

and $f(3) = (3^3 - 6 \times 3^2 + 11 \times 3 - 6) = 0.$

$\therefore \quad f(1) = f(3).$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist some $c \in]1, 3[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Rightarrow 3c^2 - 12c + 11 = 0$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 132}}{6} \Rightarrow c = \left(2 \pm \frac{1}{\sqrt{3}}\right).$$

Clearly, both the values of c lie in the interval $]1, 3[$.

Hence, Rolle's theorem is verified.

EXAMPLE 2 *Verify Rolle's theorem for the function* $f(x) = x(x-1)^2$ *in the interval* $[0, 1]$.

SOLUTION We have $f(x) = x^3 - 2x^2 + x.$

We observe here that

(i) $f(x)$ being a polynomial function, is continuous on $[0, 1]$.

(ii) $f'(x) = (3x^2 - 4x + 1)$, which clearly exists for all values of $x \in [0, 1]$.

So, $f(x)$ is differentiable on the interval $]0, 1[$.

(iii) $f(0) = 0$ and $f(1) = 0.$

$\therefore \quad f(0) = f(1).$

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist a real number $c \in]0, 1[$ such that $f'(c) = 0.$

Now, $f'(c) = 0 \Rightarrow 3c^2 - 4c + 1 = 0 \Rightarrow (c-1)(3c-1) = 0$

$$\Rightarrow c = 1 \text{ or } c = \frac{1}{3}.$$

Out of these two values, clearly $\frac{1}{3} \in]0, 1[$.

Thus, $c = \frac{1}{3} \in]0, 1[$ such that $f'(c) = 0.$

Hence, Rolle's theorem is satisfied.

EXAMPLE 3 *Verify Rolle's theorem for the function* $f(x) = (x-a)^m(x-b)^n$ *in the interval* $[a, b]$, *where m and n are positive integers.*

SOLUTION Let $f(x) = (x-a)^m(x-b)^n$, where $a \le x \le b$.

On expanding $(x-a)^m$ and $(x-b)^n$ by the binomial theorem and then taking the product, we find that $f(x)$ is a polynomial of degree $(m+n)$. But, a polynomial function being continuous everywhere, it follows that $f(x)$ is continuous in $[a, b]$.

Also, $f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1} \cdot [m(x-b) + n(x-a)]$,

which clearly exists for all $x \in]a, b[$.

$\therefore$ $f(x)$ is differentiable on $]a, b[$.

Also, $f(a) = f(b) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied. Consequently, there must exist some $c \in]a, b[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow (c-a)^{m-1}(c-b)^{n-1}[m(c-b) + n(c-a)] = 0$

$\Leftrightarrow (c-a) = 0$ or $(c-b) = 0$ or $m(c-b) + n(c-a) = 0$

$\Leftrightarrow c = a$ or $c = b$ or $c = \left(\frac{mb+na}{m+n}\right)$.

Clearly, $c = \left(\frac{mb+na}{m+n}\right) \in]a, b[$ such that $f'(c) = 0$

$\left\{\because \left(\frac{mb+na}{m+n}\right)\right.$ divides $]a, b[$ in the ratio $m:n \left.\right\}$.

Hence, Rolle's theorem is verified.

EXAMPLE 4 *Verify Rolle's theorem for each of the following functions:*

(i) $f(x) = \sin 2x$ *in* $\left[0, \frac{\pi}{2}\right]$

(ii) $f(x) = (\sin x + \cos x)$ *in* $\left[0, \frac{\pi}{2}\right]$ **[CBSE 2006]**

(iii) $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ *in* $\left[0, \frac{\pi}{2}\right]$

(iv) $f(x) = (\sin x - \sin 2x)$ *in* $[0, \pi]$

SOLUTION (i) Consider $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$.

Since the sine function is continuous at each $x \in R$, it follows that $f(x) = \sin 2x$ is continuous on $\left[0, \frac{\pi}{2}\right]$.

Also, $f'(x) = 2\cos 2x$, which clearly exists for all $x \in \left[0, \frac{\pi}{2}\right]$.

So, $f(x)$ is differentiable on $\left]0, \frac{\pi}{2}\right[$.

Also, $f(0) = f\left(\frac{\pi}{2}\right) = 0.$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist a real number $c \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Now, $f'(c) = 0 \Leftrightarrow 2\cos 2c = 0 \Leftrightarrow \cos 2c = 0$

$\Leftrightarrow 2c = \frac{\pi}{2}$, i.e., $c = \frac{\pi}{4}$.

Thus, $c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Hence, Rolle's theorem is verified.

(ii) Consider $f(x) = (\sin x + \cos x)$ in $\left[0, \frac{\pi}{2}\right]$.

By the continuity of the sine function, the cosine function and the sum of continuous functions, it follows that $f(x)$ is continuous on $\left[0, \frac{\pi}{2}\right]$.

Also, $f'(x) = (\cos x - \sin x)$, which clearly exists for all values of $x \in \left[0, \frac{\pi}{2}\right]$.

So, $f(x)$ is differentiable on $\left]0, \frac{\pi}{2}\right[$. Also, $f(0) = f\left(\frac{\pi}{2}\right) = 1.$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist some $c \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Now, $f'(c) = 0 \Leftrightarrow \cos c - \sin c = 0 \Leftrightarrow \cos c = \sin c \Leftrightarrow c = \frac{\pi}{4}$.

Thus, $c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Hence, Rolle's theorem is verified.

(iii) Consider $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ in $\left[0, \frac{\pi}{2}\right]$.

Since the cosine function is continuous everywhere, it follows that $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ is continuous on $\left[0, \frac{\pi}{2}\right]$.

Also, $f'(x) = -2\sin\left(2x - \frac{\pi}{2}\right) = 2\cos 2x$, which clearly exists for all $x \in \left]0, \frac{\pi}{2}\right[$.

$\therefore$ $f(x)$ is differentiable on $\left]0, \frac{\pi}{2}\right[$.

Further, $f(0) = \cos 2\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0.$

And, $f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0.$

$\therefore$ $f(0) = f\left(\frac{\pi}{2}\right) = 0.$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist $c \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Now, $f'(c) = 0 \Leftrightarrow 2\cos 2c = 0 \Leftrightarrow 2c = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{4}$.

Thus, $c = \frac{\pi}{4} \in \left]0, \frac{\pi}{2}\right[$ such that $f'(c) = 0.$

Hence, Rolle's theorem is verified.

(iv) Consider $f(x) = (\sin x - \sin 2x)$ in $[0, \pi]$.

Since the sine function is continuous, it follows that $g(x) = \sin x$ and $h(x) = \sin 2x$ are both continuous and so their difference is also continuous.

Consequently, $f(x) = g(x) - h(x)$ is differentiable on $[0, \pi]$.

Also, $f'(x) = (\cos x - 2\cos 2x)$, which clearly exists for all $x \in [0, \pi]$.

$\therefore$ $f(x)$ is differentiable on $]0, \pi[$.

And, $f(0) = f(\pi) = 0.$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist a real number $c \in]0, \pi[$ such that $f'(c) = 0.$

Now, $f'(c) = 0 \Leftrightarrow \cos c - 2\cos 2c = 0$

$\Leftrightarrow \cos c - 2(2\cos^2 c - 1) = 0$

$\Leftrightarrow 4\cos^2 c - \cos c - 2 = 0$

$\Leftrightarrow \cos c = \frac{1 \pm \sqrt{33}}{8} = 0.8431$ or -0.5931

$\Leftrightarrow \cos c = 0.8431$ or $\cos(180° - c) = 0.5931.$

$\Leftrightarrow c = 32°\cdot 32'$ or $c = 126°23'.$

Thus, $c \in]0, \pi[$ such that $f'(c) = 0.$

Hence, Rolle's theorem is satisfied.

EXAMPLE 5 *Verify Rolle's theorem for each of the following functions:*

(i) $f(x) = \sin^2 x$ *in* $0 \le x \le \pi$

(ii) $f(x) = e^x \cos x$ *in* $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(iii) $f(x) = \frac{\sin x}{e^x}$ *in* $0 \le x \le \pi$

SOLUTION (i) Consider $f(x) = \sin^2 x$ in $[0, \pi]$.

Let $g(x) = \sin x$ and $h(x) = x^2$.

Then, the sine function being continuous for all $x \in R$ and every polynomial function being continuous for all $x \in R$, it follows that g and h are both continuous for all $x \in R$.

But the composite of continuous functions is continuous.

$\therefore \quad (h \circ g)(x) = f(x) = \sin^2 x$ is continuous on $[0, \pi]$.

Also, $f'(x) = 2\sin x \cos x = \sin 2x$, which clearly exists for all $x \in]0, \pi[$.

So, $f(x)$ is differentiable on $]0, \pi[$.

Also, $f(0) = f(\pi) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist $c \in]0, \pi[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow \sin 2c = 0 \Leftrightarrow 2c = \pi \Leftrightarrow c = \frac{\pi}{2}$.

Thus, $c = \frac{\pi}{2} \in]0, \pi[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

(ii) Consider $f(x) = e^x \cos x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let $g(x) = e^x$ and $h(x) = \cos x$.

Then, the exponential function as well as the cosine function being continuous, it follows that $g(x) \cdot h(x) = f(x)$ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Also, $f'(x) = -e^x \sin x + e^x \cos x = e^x(\cos x - \sin x)$, which clearly exists for all $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

So, $f(x)$ is differentiable on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

Also, $f\left(-\frac{\pi}{2}\right) = e^{-\pi/2}\cos\left(-\frac{\pi}{2}\right) = 0$. And, $f\left(\frac{\pi}{2}\right) = e^{\pi/2}\cos\frac{\pi}{2} = 0$.

$\therefore \quad f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist some $c \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow e^c(\cos c - \sin c) = 0 \Leftrightarrow \cos c - \sin c = 0$

$\Leftrightarrow \sin c = \cos c \Leftrightarrow c = \frac{\pi}{4}$.

Thus, $c = \frac{\pi}{4} \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

(iii) Consider $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$.

Since $e^x \neq 0$ for any $x \in [0, \pi]$ and $f(x)$ is the quotient of two continuous functions, it follows that $f(x)$ is continuous on $[0, \pi]$.

Also, $f'(x) = \frac{e^x \cos x - e^x \sin x}{e^{2x}} = \frac{(\cos x - \sin x)}{e^x}$, which clearly exists for all $x \in]0, \pi[$.

So, $f(x)$ is differentiable on $]0, \pi[$. Also, $f(0) = f(\pi) = 0$.

So, all the conditions of Rolle's theorem are satisfied.

$\therefore$ there must exist $c \in]0, \pi[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow \frac{(\cos c - \sin c)}{e^c} = 0$

$\Leftrightarrow \cos c - \sin c = 0 \Leftrightarrow \sin c = \cos c \Leftrightarrow c = \frac{\pi}{4}$.

Thus, $c = \frac{\pi}{4} \in]0, \pi[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

EXAMPLE 6 *Verify Rolle's theorem for the function* $f(x) = x(x + 3)e^{-(x/2)}$ *in* $[-3, 0]$.

SOLUTION Since a polynomial function as well as an exponential function is continuous and the product of two continuous functions is continuous, it follows that $f(x)$ is continuous on the given interval $[-3, 0]$.

Now, $f'(x) = (2x + 3)e^{-(x/2)} - \frac{1}{2}e^{-(x/2)}(x^2 + 3x)$

$= e^{-(x/2)}\left(\frac{x + 6 - x^2}{2}\right)$,

which is clearly finite for all values of x in $]-3, 0[$.

So, $f(x)$ is differentiable on $]-3, 0[$. Also, $f(-3) = f(0) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist $c \in]-3, 0[$ such that $f'(c) = 0$.

But, $f'(c) = 0 \Leftrightarrow e^{-(c/2)}\left(\frac{c + 6 - c^2}{2}\right) = 0 \Leftrightarrow c + 6 - c^2 = 0$

$\Leftrightarrow (3 - c)(c + 2) = 0 \Leftrightarrow c = 3$ or $c = -2$.

Thus, $c = -2 \in]-3, 0[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

EXAMPLE 7 *Verify Rolle's theorem for the function* $f(x) = \{\log (x^2 + 2) - \log 3\}$ *on* $[-1, 1]$.

SOLUTION Clearly, $f(x) = \{\log (x^2 + 2) - \log 3\}$ has a unique and definite value for each $x \in [-1, 1]$. So, at each point of $[-1, 1]$, the limit of the function is equal to the value of the function.

So, $f(x)$ is continuous on $[-1, 1]$.

Also, $f'(x) = \dfrac{2x}{(x^2 + 2)}$, which is clearly finite for each $x \in [-1, 1]$.

So, $f(x)$ is differentiable on $]-1, 1[$.

Also, $f(-1) = f(1) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist $c \in]-1, 1[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow \dfrac{2c}{(c^2 + 2)} = 0 \Leftrightarrow 2c = 0$, i.e., $c = 0$.

Thus, $c = 0 \in]-1, 1[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

EXAMPLE 8 *Verify Rolle's theorem for the following functions:*

(i) $f(x) = \sqrt{4 - x^2}$ *in* $[-2, 2]$

(ii) $f(x) = \log \left[\dfrac{x^2 + ab}{(a + b)x} \right]$ *in* $[a, b]$, *where* $0 < a < b$

SOLUTION (i) Consider $f(x) = \sqrt{4 - x^2}$ in $[-2, 2]$.

Clearly, $f(x) = \sqrt{4 - x^2}$ has a unique and definite value for each $x \in [-2, 2]$. So, at each point of $[-2, 2]$, the limit is equal to the value of the function.

$\therefore$ $f(x)$ is continuous on $[-2, 2]$.

Also, $f'(x) = \dfrac{-x}{\sqrt{4 - x^2}}$, which clearly exists for each $x \in [-2, 2]$.

So, $f(x)$ is differentiable on $]-2, 2[$.

Also, $f(-2) = f(2) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist $c \in]-2, 2[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow \dfrac{-c}{\sqrt{4 - c^2}} = 0 \Leftrightarrow c = 0$.

Thus, $c = 0 \Leftrightarrow]-2, 2[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

(ii) Consider

$$f(x) = \log\left[\frac{x^2 + ab}{(a+b)x}\right] = \log(x^2 + ab) - \log(a+b) - \log x.$$

Clearly, $f(x)$ has a unique and definite value for each $x \in [a, b]$, where $0 < a < x < b$.

So, the value of the function is equal to the limit of the function at each point of $[a, b]$.

$\therefore$ $f(x)$ is continuous on $[a, b]$.

Also, $f'(x) = \left[\frac{2x}{(x^2 + ab)} - \frac{1}{x}\right]$, which is clearly definite and finite for all values of x in $]a, b[$.

So, $f(x)$ is differentiable on $]a, b[$. Moreover, $f(a) = f(b) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist $c \in]a, b[$ such that $f'(c) = 0$.

But, $f'(c) = 0 \Leftrightarrow \frac{2c}{(c^2 + ab)} - \frac{1}{c} = 0$

$$\Leftrightarrow \frac{(c^2 - ab)}{c(c^2 + ab)} = 0 \Leftrightarrow c^2 - ab = 0 \Leftrightarrow c = \sqrt{ab}.$$

Now, c being the geometric mean of a and b, it follows that $a < c < b$.

Thus, $c \in]a, b[$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

EXAMPLE 9 *Verify Rolle's theorem for the function* $f(x) = 2x^3 + x^2 - 4x - 2$.

SOLUTION Since a polynomial function is everywhere continuous and differentiable, the given function is continuous as well as differentiable on every interval.

To identify the interval, we solve the equation $f(x) = 0$.

Now, $f(x) = 0 \Leftrightarrow 2x^3 + x^2 - 4x - 2 = 0$

$$\Leftrightarrow (x^2 - 2)(2x + 1) = 0 \Leftrightarrow x^2 = 2 \text{ or } x = -\frac{1}{2}$$

$$\Leftrightarrow x = \sqrt{2} \text{ or } x = -\sqrt{2} \text{ or } x = -\frac{1}{2}.$$

So, we consider the given function in $[-\sqrt{2}, \sqrt{2}]$.

Clearly, $f(-\sqrt{2}) = f(\sqrt{2}) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist $c \in]-\sqrt{2}, \sqrt{2}[$ such that $f'(c) = 0$.

But, $f'(x) = (6x^2 + 2x - 4)$.

$\therefore$ $f'(c) = 0 \Leftrightarrow 6c^2 + 2c - 4 = 0 \Leftrightarrow 2(3c - 2)(c + 1) = 0$

$\Leftrightarrow c = 2/3$ or $c = -1$.

Clearly, both these points lie in $[-\sqrt{2}, \sqrt{2}]$.

Hence, Rolle's theorem is verified.

EXAMPLE 10 *Discuss the applicability of Rolle's theorem to the functions:*
(i) $f(x) = x^2$ *in* $[1, 2]$ *(ii)* $f(x) = x^{2/3}$ *in* $[-1, 1]$ **[CBSE 1999]**

SOLUTION (i) Consider $f(x) = x^2$ in $[1, 2]$.

Since a polynomial function is continuous and differentiable everywhere, the first two conditions of Rolle's theorem are satisfied.

But, $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$. So, $f(1) \neq f(2)$.

And, therefore, the condition $f(a) = f(b)$ is violated.

Hence, Rolle's theorem is not applicable for $f(x) = x^2$ in $[1, 2]$.

(ii) Consider $f(x) = x^{2/3}$ in $[-1, 1]$.

We have

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{2/3} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}} = \infty.$$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{(-h)^{2/3} - 0}{-h} = \lim_{h\to 0} \frac{-1}{(h)^{1/3}} = -\infty.$$

$\therefore$ $Rf'(0) \neq Lf'(0)$, i.e., $f'(0)$ does not exist.

Thus, $f(x)$ is not differentiable at $x = 0 \in]-1, 1[$.

So, the condition of differentiability at each point of the given interval is not satisfied.

Hence, Rolle's theorem is not applicable to $f(x) = x^{2/3}$ in $[-1, 1]$.

EXAMPLE 11 *Discuss the applicability of Rolle's theorem on:*
(i) $f(x) = |x|$ *in* $[-1, 1]$ *(ii)* $f\ (x) = \tan x$ *in* $[0, \pi]$

SOLUTION (i) Consider $f(x) = |x|$ in $[-1, 1]$.

We may express it as $f(x) = \begin{cases} -x & \text{when } -1 \le x < 0 \\ x & \text{when } 0 \le x \le 1 \end{cases}$

Clearly, $f(-1) = f(1) = 1$.

But, $Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$

And,

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1.$$

$\therefore$ $Rf'(0) \neq Lf'(0)$.

This shows that $f(x)$ is not differentiable at $x = 0$.

Thus, the condition of differentiability at each point of the given interval is not satisfied.

(ii) Consider $f(x) = \tan x$ in $[0, \pi]$.

Clearly, $f(0) = f(\pi) = 0$. But, $\tan \frac{\pi}{2}$ does not have a definite value.

So, $f(x) = \tan x$ is not continuous at $x = \frac{\pi}{2}$.

Hence, the condition of continuity at each point of the given interval is violated.

EXAMPLE 12 *Discuss the applicability of Rolle's theorem on the function*

$$f(x) = \begin{cases} (x^2 + 1), & \text{when } 0 \le x \le 1 \\ (3 - x), & \text{when } 1 < x \le 2. \end{cases}$$

SOLUTION The given function has been defined in [0, 2].

And, $f(0) = f(2) = 1$.

Now, we consider the differentiability of $f(x)$ at $x = 1$.

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[3 - (1+h)] - 2}{h} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{(1-h)^2 + 1 - 2}{-h} = \lim_{h \to 0} (2 - h) = 2.$$

Thus, $Rf'(1) \neq Lf'(1)$.

So, $f(x)$ is not differentiable at $x = 1 \in]0, 2[$.

Thus, the condition of differentiability at each point of the given interval is not satisfied.

EXAMPLE 13 *If Rolle's theorem holds for the function* $f(x) = x^3 + bx^2 + ax + 5$ *on* $[1, 3]$ *with* $c = \left(2 + \frac{1}{\sqrt{3}}\right)$, *find the values of a and b.*

SOLUTION We have, $f'(x) = 3x^2 + 2bx + a$.

$$\therefore \quad f'(c) = 0 \Leftrightarrow 3c^2 + 2bc + a = 0$$

$$\Leftrightarrow \quad c = \frac{-2b \pm \sqrt{4b^2 - 12a}}{6} = \frac{-b \pm \sqrt{b^2 - 3a}}{3}.$$

$$\text{Now, } c = \left(2 + \frac{1}{\sqrt{3}}\right) \Leftrightarrow \frac{-b + \sqrt{b^2 - 3a}}{3} = \left(2 + \frac{1}{\sqrt{3}}\right)$$

$$\Leftrightarrow \frac{-b}{3} = 2 \text{ and } \frac{\sqrt{b^2 - 3a}}{3} = \frac{1}{\sqrt{3}}$$

$$\Leftrightarrow b = -6 \text{ and } b^2 - 3a = 3 \Leftrightarrow b = -6 \text{ and } a = 11.$$

Hence, $a = 11$ and $b = -6$.

EXAMPLE 14 *At what points on the curve* $y = (\cos x - 1)$ *in* $[0, 2\pi]$, *is the tangent parallel to the x-axis?*

SOLUTION Consider $f(x) = (\cos x - 1)$ in $[0, 2\pi]$.

Now, the cosine function being continuous and the constant function being continuous, it follows that their difference $(\cos x - 1)$ is continuous. So, $f(x)$ is continuous on $[0, 2\pi]$.

$f'(x) = -\sin x$, which exists for all $x \in]0, 2\pi[$.

$\therefore$ $f(x)$ is differentiable on $]0, 2\pi[$.

Also, $f(0) = f(2\pi) = 0$.

So, all the conditions of Rolle's theorem are satisfied.

So, there must exist some $c \in]0, 2\pi[$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Leftrightarrow -\sin c = 0 \Leftrightarrow \sin c = 0$

$\Leftrightarrow c = 0$ or $c = \pi$ or $c = 2\pi$.

Thus, the tangent to the curve is parallel to the x-axis at each of the points $x = 0$, $x = \pi$ and $x = 2\pi$.

Now, $x = 0$ and $y = \cos x - 1 \Rightarrow y = 0$;

$x = \pi$ and $y = \cos x - 1 \Rightarrow y = -2$;

$x = 2\pi$ and $y = \cos x - 1 \Rightarrow y = 0$.

$\therefore$ the required points are $(0, 0)$, $(\pi, -2)$ and $(2\pi, 0)$.

EXERCISE 11C

Verify Rolle's theorem for each of the following functions:

1. $f(x) = x^2$ on $[-1, 1]$
2. $f(x) = x^2 - x - 12$ in $[-3, 4]$ **[CBSE 2001]**
3. $f(x) = x^2 - 5x + 6$ in $[2, 3]$ **[CBSE 2002C]**
4. $f(x) = x^2 - 3x - 18$ in $[-3, 6]$
5. $f(x) = x^2 - 4x + 3$ in $[1, 3]$ **[CBSE 2007]**
6. $f(x) = x(x-4)^2$ in $[0, 4]$
7. $f(x) = x^3 - 7x^2 + 16x - 12$ in $[2, 3]$
8. $f(x) = x^3 + 3x^2 - 24x - 80$ in $[-4, 5]$
9. $f(x) = (x-1)(x-2)(x-3)$ in $[1, 3]$
10. $f(x) = (x-1)(x-2)^2$ in $[1, 2]$ **[CBSE 2006]**
11. $f(x) = (x-2)^4(x-3)^3$ in $[2, 3]$
12. $f(x) = \sqrt{1-x^2}$ in $[-1, 1]$
13. $f(x) = \cos x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
14. $f(x) = \cos 2x$ in $[0, \pi]$
15. $f(x) = \sin 3x$ in $[0, \pi]$
16. $f(x) = \sin x + \cos x$ in $\left[0, \frac{\pi}{2}\right]$
17. $f(x) = e^{-x} \sin x$ in $[0, \pi]$
18. $f(x) = e^{-x}(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$
19. $f(x) = \sin x - \sin 2x$ in $[0, 2\pi]$
20. $f(x) = x(x+2)e^x$ in $[-2, 0]$
21. Show that $f(x) = x(x-5)^2$ satisfies Rolle's theorem on $[0, 5]$ and that the value of c is $(5/3)$.

Discuss the applicability of Rolle's theorem, when:

22. $f(x) = (x-1)(2x-3)$, where $1 \le x \le 3$
23. $f(x) = x^{1/2}$ on $[-1, 1]$
24. $f(x) = 2 + (x-1)^{2/3}$ on $[0, 2]$

25. $f(x) = \cos\frac{1}{x}$ on $[-1, 1]$

26. $f(x) = [x]$ on $[-1, 1]$, where $[x]$ denotes the greatest integer not exceeding x

27. Using Rolle's theorem, find the point on the curve $y = x(x-4)$, $x \in [0, 4]$, where the tangent is parallel to the x-axis. **[CBSE 2000]**

ANSWERS (EXERCISE 11C)

22. Not applicable, since $f(1) \neq f(3)$

23. Not applicable, since $f'(0)$ does not exist

24. Not applicable, since $f'(1)$ does not exist

25. Not applicable, since $f(x)$ is discontinuous at $x = 0$

26. Not applicable, since $f(x)$ is discontinuous at $x = 0$

27. $(2, -4)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11C)

9. $f(x) = (x^3 - 6x^2 + 11x - 6)$.

22. $f(1) \neq f(3)$.

23. $f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(0)$ does not exist, where $0 \in [-1, 1]$.

24. $f'(1)$ does not exist.

25. $f(x)$ is discontinuous at $x = 0$.

26. $f(x)$ is not continuous at $x = 0$.

Lagrange's Mean-value Theorem

Let f be a real function such that

(i) $f(x)$ is continuous on $[a, b]$, *(ii) $f(x)$ is differentiable on $]a, b[$.*

Then, there exists a real number $c \in]a, b[$ such that $f'(c) = \frac{f(b) - f(a)}{(b-a)}$.

PROOF Let $F(x) = f(x) + Ax$, where A is a constant to be chosen in such a way that $F(a) = F(b)$.

Now, $F(a) = F(b) \Leftrightarrow f(a) + Aa = f(b) + Ab \Leftrightarrow A = \frac{f(b) - f(a)}{(a-b)}$.

$\therefore \quad F(x) = f(x) + \frac{f(b) - f(a)}{(a-b)} \cdot x \quad \ldots$ (i)

Now, $f(x)$ is continuous on $[a, b]$, and $\left\{\frac{f(b) - f(a)}{(a-b)} \cdot x\right\}$ being a polynomial function, is continuous on $[a, b]$.

And, the sum of two continuous functions being continuous, it follows from (i) that $F(x)$ is continuous.

Also, $f(x)$ is differentiable on $]a, b[$, and since the polynomial function $\frac{f(b)-f(a)}{(a-b)}x$ is differentiable on $]a, b[$, it follows that $F(x)$ is differentiable on $]a, b[$.

Also, $F(a) = F(b)$.

Thus, all the conditions of Rolle's theorem are satisfied by $F(x)$.

So, there must exist some $c \in]a, b[$ such that $F'(c) = 0$.

Now, $F(x) = f(x) + \frac{f(b)-f(a)}{(a-b)} \cdot x \Rightarrow F'(x) = f'(x) + \frac{f(b)-f(a)}{(a-b)}$.

$\therefore \quad F'(c) = 0 \Leftrightarrow f'(c) + \frac{f(b)-f(a)}{(a-b)} = 0 \Leftrightarrow f'(c) = \frac{f(b)-f(a)}{(b-a)}$.

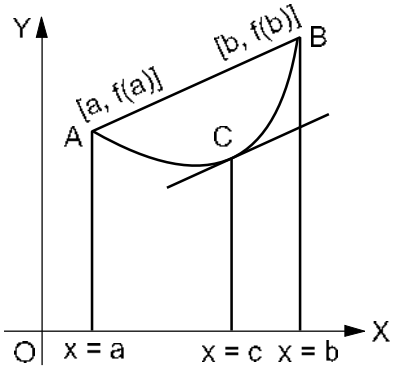
GEOMETRICAL SIGNIFICANCE OF THE MEAN-VALUE THEOREM Let $y = f(x)$ be a given function defined on $[a, b]$, which is continuous on $[a, b]$ and differentiable on $]a, b[$.

Then, by Lagrange's mean-value theorem, there exists some $c \in]a, b[$ such that

$$f'(c) = \frac{f(b)-f(a)}{(b-a)} \quad \ldots \text{(i)}$$

Now, if we draw the curve $y = f(x)$ and take the points $A[a, f(a)]$ and $B[b, f(b)]$ on the curve then

$$\text{slope of chord } AB = \frac{f(b)-f(a)}{(b-a)} \quad \ldots \text{(ii)}$$



Thus, from (i) and (ii), we have

$f'(c) =$ slope of chord AB.

This shows that *the tangent to the curve $y = f(x)$ at the point $x = c$ is parallel to the chord AB.*

PHYSICAL SIGNIFICANCE OF THE MEAN-VALUE THEOREM Let a particle be moving in a straight line and let $f(a)$ and $f(b)$ be its positions from the starting point, at the times a and b respectively.

Then, average speed of the particle $= \frac{f(b)-f(a)}{(b-a)}$.

We also know that $f'(c)$ denotes the instantaneous speed of the particle at time c.

Now, by the mean-value theorem, $f'(c) = \frac{f(b)-f(a)}{(b-a)}$.

So, *the mean-value theorem says that at some time c between a and b, the instantaneous speed of the particle would be equal to the average speed.*

SOLVED EXAMPLES

EXAMPLE 1 *Verify Lagrange's mean-value theorem for the following functions:*

(i) $f(x) = x(2-x)$ *in* $[0, 1]$

(ii) $f(x) = x(x+4)^2$ *in* $[0, 4]$

(iii) $f(x) = x + \frac{1}{x}$ *in* $[1, 3]$

(iv) $f(x) = (x-1)(x-2)(x-3)$ *in* $[0, 4]$

(v) $f(x) = \sqrt{x^2-4}$ **[CBSE 2002]**

SOLUTION (i) Consider $f(x) = x(2-x)$ in $[0, 1]$.

The given function is $f(x) = 2x - x^2$.

It, being a polynomial function, is continuous on $[0, 1]$.

Also, $f'(x) = 2 - 2x$, which exists for all x in $[0, 1]$.

So, $f(x)$ is differentiable on $]0, 1[$.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in]0, 1[$ such that

$$f'(c) = \frac{f(1)-f(0)}{(1-0)} = 1.$$

Now, $f'(c) = 1 \Leftrightarrow 2 - 2c = 1 \Leftrightarrow c = \frac{1}{2} \in]0, 1[$.

Thus, $c = \frac{1}{2} \in]0, 1[$ such that $f'(c) = \frac{f(1)-f(0)}{1-0}$.

Hence, Lagrange's mean-value theorem is verified.

(ii) Consider $f(x) = x(x+4)^2$ in $[0, 4]$.

The given function is $f(x) = x^3 + 8x^2 + 16x$.

It, being a polynomial function, is continuous on $[0, 4]$.

Also, $f'(x) = (3x^2 + 16x + 16)$, which exists for all $x \in]0, 4[$.

So, $f(x)$ is differentiable on $]0, 4[$.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in]0, 4[$ such that

$$f'(c) = \frac{f(4)-f(0)}{(4-0)} = \left(\frac{256-0}{4}\right) = 64.$$

Now, $f'(c) = 64 \Leftrightarrow 3c^2 + 16c + 16 = 64$

$\Leftrightarrow 3c^2 + 16c - 48 = 0 \Leftrightarrow c = \frac{-8 \pm 4\sqrt{13}}{3}$.

Clearly, $c = \frac{-8 + 4\sqrt{13}}{3} = 2.13 \in]0, 4[$.

Hence, Lagrange's mean-value theorem is verified.

(iii) Consider $f(x) = x + \frac{1}{x}$ in [1, 3]. The given function is

$$f(x) = \frac{x^2 + 1}{x}.$$

Since it is a rational function such that $x \neq 0$, it is continuous in [1, 3].

Also, $f'(x) = \left(1 - \frac{1}{x^2}\right) = \left(\frac{x^2 - 1}{x^2}\right)$, which clearly exists for all values of x in]1, 3[.

So, $f(x)$ is differentiable on]1, 3[.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in]1, 3[$ such that

$$f'(c) = \frac{f(3) - f(1)}{(3 - 1)} = \frac{\left(\frac{10}{3} - 2\right)}{2} = \frac{2}{3}.$$

Now, $f'(c) = \frac{2}{3} \Leftrightarrow \frac{c^2 - 1}{c^2} = \frac{2}{3} \Leftrightarrow c = \sqrt{3} \in]1, 3[$.

Thus, $c = \sqrt{3} \in]1, 3[$ such that $f'(c) = \frac{f(3) - f(1)}{(3 - 1)}$.

Hence, Lagrange's mean-value theorem is verified.

(iv) Consider $f(x) = (x - 1)(x - 2)(x - 3)$ in [0, 4].

The given function is $f(x) = x^3 - 6x^2 + 11x - 6$.

Being a polynomial function, it is continuous on [0, 4].

Also, $f'(x) = (3x^2 - 12x + 11)$, which exists for all $x \in]0, 4[$.

So, $f(x)$ is differentiable on]0, 4[.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there exists some $c \in]0, 4[$ such that

$$f'(c) = \frac{f(4) - f(0)}{(4 - 0)} = \frac{6 - (-6)}{4} = 3.$$

Now, $f'(c) = 3 \Leftrightarrow 3c^2 - 12c + 11 = 3 \Leftrightarrow 3c^2 - 12c + 8 = 0$

$$\Leftrightarrow c = \frac{12 \pm \sqrt{144 - 96}}{6} = \frac{6 \pm 2\sqrt{3}}{3}.$$

$$\Leftrightarrow c = 3.155 \text{ or } c = 0.845 \quad [\because \sqrt{3} = 1.732].$$

Clearly, both these values of c lie in]0, 4[.

Hence, Lagrange's mean-value theorem is verified.

(v) Consider $f(x) = \sqrt{x^2 - 4}$ in [2, 4].

Clearly, $f(x)$ has a definite and unique value for each $x \in [2, 4]$. So, at every point [2, 4], the value of $f(x)$ is equal to the limit of $f(x)$.

So, $f(x)$ is continuous on [2, 4].

Also, $f'(x) = \dfrac{x}{\sqrt{x^2-4}}$, which exists for all $x \in]2, 4[$.

So, $f(x)$ is differentiable on]2, 4[.

So, there must exist some $c \in]2, 4[$ such that

$$f'(c) = \frac{f(4)-f(2)}{(4-2)} = \frac{\sqrt{12}-0}{2} = \sqrt{3}.$$

Now,

$$f'(c) = \sqrt{3} \Leftrightarrow \frac{c}{\sqrt{c^2-4}} = \sqrt{3} \Leftrightarrow c^2 = 3(c^2-4) \Leftrightarrow c = \pm\sqrt{6}.$$

Clearly, $c = \sqrt{6} \in]2, 4[$ such that $f'(c) = \dfrac{f(4)-f(2)}{(4-2)}$.

Hence, Lagrange's mean-value theorem is verified.

EXAMPLE 2 *Verify the hypothesis and conclusion of Lagrange's mean-value theorem for the function* $f(x) = \dfrac{1}{(4x-1)}$, $1 \le x \le 4$.

SOLUTION Clearly, for each $x \in [1, 4]$, $f(x)$ has a definite and unique value.

So, $f(x)$ is continuous on [1, 4].

Also, $f'(x) = \dfrac{-4}{(4x-1)^2}$, which exists for all $x \in]1, 4[$.

So, $f(x)$ is differentiable on]1, 4[.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in]1, 4[$ such that

$$f'(c) = \frac{f(4)-f(1)}{(4-1)} = \frac{1}{3}\left(\frac{1}{15} - \frac{1}{3}\right) = \frac{-4}{45}.$$

Now, $f'(c) = \dfrac{-4}{45} \Leftrightarrow \dfrac{-4}{(4c-1)^2} = \dfrac{-4}{45} \Leftrightarrow (4c-1)^2 = 45$

$$\Leftrightarrow (4c-1) = \pm 3\sqrt{5} \Leftrightarrow c = \left(\frac{1 \pm 3\sqrt{5}}{4}\right).$$

Clearly, $c = \dfrac{1+3\sqrt{5}}{4} = \left(\dfrac{1+3\times 2.23}{4}\right) = 1.92 \in]1, 4[$.

Hence, Lagrange's mean-value theorem is verified.

EXAMPLE 3 *Find 'c' of the mean-value theorem for the functions:*

(i) $f(x) = 2x^2 - 10x + 29$ *in* [2, 7]

(ii) $f(x) = x(x-1)(x-2)$ *in* $\left[0, \dfrac{1}{2}\right]$

SOLUTION (i) $f(x) = 2x^2 - 10x + 29 \Leftrightarrow f'(x) = 4x - 10.$

Now, $f'(c) = \dfrac{f(7) - f(2)}{(7-2)} \Leftrightarrow (4c - 10) = \dfrac{(57 - 17)}{5}$

$\Leftrightarrow 4c = 18 \Leftrightarrow c = 4.5 \in]2, 7[.$

$\therefore \quad c = 4.5.$

(ii) The given function is $f(x) = x^3 - 3x^2 + 2x.$

$\therefore \quad f'(x) = 3x^2 - 6x + 2.$

Now,

$$f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\left(\frac{1}{2} - 0\right)} \Leftrightarrow 3c^2 - 6c + 2 = \frac{\left(\frac{3}{8} - 0\right)}{\left(\frac{1}{2}\right)}$$

$$\Leftrightarrow 12c^2 - 24c + 5 = 0$$

$$\Leftrightarrow c = \frac{24 \pm \sqrt{576 - 240}}{24} = \frac{6 \pm \sqrt{21}}{6}.$$

Clearly, $c = \dfrac{6 - \sqrt{21}}{6} = \dfrac{6 - 4.58}{6} = 0.24 \in \left[0, \dfrac{1}{2}\right].$

Hence, $c = 0.24.$

EXAMPLE 4 *Using Lagrange's mean-value theorem, find a point on the curve* $y = \sqrt{x - 2}$, *defined in the interval* [2, 3], *where the tangent is parallel to the chord joining the end points of the curve.*

SOLUTION Let $f(x) = \sqrt{x - 2}.$

Clearly, for each $x \in [2, 3]$, the function $f(x)$ has a definite and unique value.

So, $f(x)$ is continuous on [2, 3].

Also, $f'(x) = \dfrac{1}{2\sqrt{x - 2}}$, which exists for all x in]2, 3[.

So, $f(x)$ is differentiable on]2, 3[.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in]2, 3[$ such that $f'(c) = \dfrac{f(3) - f(2)}{(3-2)} = 1.$

Now, $f'(c) = 1 \Leftrightarrow \dfrac{1}{2\sqrt{c-2}} = 1 \Leftrightarrow c - 2 = \dfrac{1}{4}$, i.e., $c = \dfrac{9}{4} \in]2, 3[.$

Now, $x = \dfrac{9}{4}$ and $y = \sqrt{x - 2} \Leftrightarrow y = \sqrt{\left(\dfrac{9}{4} - 2\right)} = \dfrac{1}{2}.$

$\therefore$ the tangent to the given curve at the point $\left(\dfrac{9}{4}, \dfrac{1}{2}\right)$ is parallel to the chord joining the end points of the curve.

EXAMPLE 5 *Find a point on the parabola* $y = (x-3)^2$, *where the tangent is parallel to the chord joining* $(3, 0)$ *and* $(4, 1)$. **[CBSE 2000C]**

SOLUTION Let us apply Lagrange's mean-value theorem for the function $f(x) = (x-3)^2$ in the interval $[3, 4]$.

Now, $f(x)$ being a polynomial function, it is continuous on $[3, 4]$.

Also, $f'(x) = 2(x-3)$, which exists for all $x \in]3, 4[$.

So, $f(x)$ is differentiable on $]3, 4[$.

Thus, both the conditions of Lagrange's mean-value theorem are satisfied.

So, there must exist a point $c \in]3, 4[$ such that

$$f'(c) = \frac{f(4)-f(3)}{(4-3)} = 1.$$

Now, $f'(c) = 1 \Leftrightarrow 2(c-3) = 1 \Leftrightarrow c = \frac{7}{2} \in]3, 4[$.

Now, $x = \frac{7}{2}$ and $y = (x-3)^2 \Leftrightarrow y = \frac{1}{4}$.

Thus, at the point $\left(\frac{7}{2}, \frac{1}{4}\right)$ on the given curve the tangent is parallel to the chord joining (3, 0) and (4, 1).

EXERCISE 11D

Verify Lagrange's mean-value theorem for each of the following functions:

1. $f(x) = x^2 + 2x + 3$ on $[4, 6]$ **[CBSE 2006]**
2. $f(x) = x^2 + x - 1$ on $[0, 4]$ **[CBSE 2002]**
3. $f(x) = 2x^2 - 3x + 1$ on $[1, 3]$
4. $f(x) = x^3 + x^2 - 6x$ on $[-1, 4]$
5. $f(x) = (x-4)(x-6)(x-8)$ on $[4, 10]$
6. $f(x) = e^x$ on $[0, 1]$
7. $f(x) = x^{2/3}$ on $[0, 1]$
8. $f(x) = \log x$ on $[1, e]$
9. $f(x) = \tan^{-1} x$ on $[0, 1]$
10. $f(x) = \sin x$ on $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$
11. $f(x) = (\sin x + \cos x)$ on $\left[0, \frac{\pi}{2}\right]$
12. Show that Lagrange's mean-value theorem is not applicable to $f(x) = |x|$ on $[-1, 1]$.
13. Show that Lagrange's mean-value theorem is not applicable to $f(x) = \frac{1}{x}$ on $[-1, 1]$.

14. Find 'c' of Lagrange's mean-value theorem for

(i) $f(x) = (x^3 - 3x^2 + 2x)$ on $\left[0, \frac{1}{2}\right]$

(ii) $f(x) = \sqrt{25 - x^2}$ on [1, 5]

(iii) $f(x) = \sqrt{x + 2}$ on [4, 6]

15. Using Lagrange's mean-value theorem, find a point on the curve $y = x^2$, where the tangent is parallel to the line joining the points (1, 1) and (2, 4).

[CBSE 2000C]

16. Find a point on the curve $y = x^3$, where the tangent to the curve is parallel to the chord joining the points (1, 1) and (3, 27).

17. Find the points on the curve $y = x^3 - 3x$, where the tangent to the curve is parallel to the chord joining (1, –2) and (2, 2).

18. If $f(x) = x(1 - \log x)$, where $x > 0$, show that
$(a - b)\log c = b(1 - \log b) - a(1 - \log a)$, where $0 < a < c < b$.

ANSWERS (EXERCISE 11D)

14. (i) $1 - \sqrt{\frac{7}{12}}$ (ii) $\sqrt{15}$ (iii) 4.964 **15.** $\left(\frac{3}{2}, \frac{9}{4}\right)$

16. $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$ **17.** $\left(\sqrt{\frac{7}{3}}, \frac{-2}{3}\sqrt{\frac{7}{3}}\right), \left(-\sqrt{\frac{7}{3}}, \frac{2}{3}\sqrt{\frac{7}{3}}\right)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11D)

6. $c = \log(e - 1)$. 8. $2 < e < 3$. 12. $f'(0)$ does not exist.

13. $f(x)$ is discontinuous at $x = 0$.

18. $f(x)$ is continuous for all $x > 0$ and $f'(x) = -\log x$, which exists for all $x > 0$.

$\therefore \quad f'(c) = \frac{f(b) - f(a)}{(b - a)} \Rightarrow -\log c = \frac{b(1 - \log b) - a(1 - \log a)}{(b - a)}$.